CHAPTER V.

THE RISE OF MODERN MATHEMATICS—THE INFINITESIMAL CALCULUS.

In the third chapter we have seen that the ancient Greeks were sometimes occupied with the theoretically exact determination of the areas enclosed by curvilinear figures, and that they used the "method of exhaustion," and, to demonstrate the results which they got, an indirect method. We have seen, too, a "method of indivisibles," which was direct and seemed to gain in brevity and efficiency from a certain lack of correctness in expression and perhaps even a small inexactness in thought. We shall find the same merits and demerits—both, especially the merits, intensified—in the "infinitesimal calculus."

By the side of researches on quadratures and the finding of volumes and centres of gravity developed the methods of drawing tangents to curves. We have begun to deal with this subject in the third chapter: here we shall illustrate the considerations of Fermat (1601–1665) and Barrow (1630–1677)—the intellectual descendants of Kepler—by a simple example.

Let it be proposed to draw a tangent at a given point $P$ in the circumference of a circle of centre $O$ and equation $x^2 + y^2 = 1$. Let us take the circle to be a polygon of a great number of sides; let $PQ$ be one of these sides, and produce it to meet the $x$-axis at $T$. Then $PT$ will be the tangent in question.
Let the co-ordinates of P be X and Y; those of Q will be \(X+e\) and \(Y+a\), where \(e\) and \(a\) are infinitely small increments, positive or negative. From a figure in which the ordinates and abscissae of P and Q are drawn, so that the ordinate of P is \(PR\), we can see, by a well-known property of triangles, that \(TR\) is to \(RP\) (or \(Y\)) as \(e\) is to \(a\). Now, \(X\) and \(Y\) are related by the equation \(X^2+Y^2=1\), and, since Q is also on the locus \(x^2+y^2=1\), we have \((X+e)^2+(Y+a)^2=1\). From the two equations in which \(X\) and \(Y\) occur, we conclude that

\[2eX+e^2+2aY+a^2=0, \text{ and hence } \frac{e}{a}(X+\frac{e}{2})+Y+\frac{a}{2}=0.\]

But \(\frac{e}{a}=\frac{TR}{Y}\); hence \(TR=-\frac{Y(Y+\frac{a}{2})}{X+\frac{e}{2}}\). Now, \(a\) and \(e\) may be neglected in comparison with \(X\) and \(Y\), and thus we can say that, at any rate very nearly, we have \(TR=-\frac{Y^2}{X}\). But this is exactly right, for, since \(TP\) is at right angles to \(OP\), we know that \(OR\) is to \(RP\) as \(PR\) is to \(RT\). Here \(X\) and \(Y\) are constant, but we can say that the abscissa of the point where the tangent at any point (say \(y\)) of the circle cuts the \(x\)-axis is given by adding \(-\frac{y^2}{x}\) to \(x\).

Thus, we can find tangents by considering the ratios of infinitesimals to one another. The method obviously applies to other curves besides circles; and Barrow’s method and nomenclature leads us straight to the notation and nomenclature of Leibniz. Barrow called the triangle \(PQS\), where \(S\) is where a parallel to the \(x\)-axis through \(Q\) meets \(PR\), the “differential triangle,” and Leibniz denoted Barrow’s \(a\) and \(e\) by \(dy\) and \(dx\) (short for the “differential of \(y\)” and “the differential of \(x\),” so that “\(d\)” does not denote a number but “\(dx\)” altogether stands for
an "infinitesimal") respectively, and called the collection of rules for working with his signs the "differential calculus."

But before the notation of the differential calculus and the rules of it were discovered by Gottfried Wilhelm von Leibniz (1646-1716), the celebrated German philosopher, statesman, and mathematician, he had invented the notation and found some of the rules of the "integral calculus": thus, he had used the now well-known sign "\( f \)" or long "s" as short for "the sum of," when considering the sum of an infinity of infinitesimal elements as we do in the method of indivisibles. Suppose that we propose to determine the area included between a certain curve \( y = f(x) \), the x-axis, and two fixed ordinates whose equations are \( x = a \) and \( x = b \); then, if we make use of the idea and notation of differentials, we notice that the area in question can be written as

\[ "fy \cdot dx," \]

the summation extending from \( x = a \) to \( x = b \). We will not here further concern ourselves about these boundaries. Notice that in the above expression we have put a dot between the "\( y \)" and the "\( dx \)": this is to indicate that \( y \) is to multiply \( dx \). Hitherto we have used juxtaposition to denote multiplication, but here \( d \) is written close to \( x \) with another end in view; and it is desirable to emphasise the difference between "\( d \)" used in the sense of an adjective and "\( d \)" used in the sense of a multiplying number, at least until the student can easily tell the difference by the context. If, then, we imagine the abscissa divided into equal infinitesimal parts, each of length \( dx \), corresponding to the constituents called "points" in the method of indivisibles, \( y \cdot dx \) is the area of the little rectangle of sides \( dx \) and \( y \) which stand at the end of the abscissa \( x \). If, now, instead of extending to
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$x=b$, the summation extends to the ordinate at the indeterminate or "variable" point $x$, $y \cdot dx$ becomes a function of $x$.

Now, if we think what must be the differential of this sum—the infinitesimal increment that it gets when the abscissa of length $x$, which is one of the boundaries, is increased by $dx$—we see that it must be $y \cdot dx$. Hence

$$d(fy \cdot dx) = y \cdot dx,$$

and hence the sign "$d$" destroys, so to speak, the effect of the sign "$f$". We also have $\int dx = x$, and find that this summation is the inverse process to differentiation. Thus the problems of tangents and quadratures are inverses of one another. This vital discovery seems to have been first made by Barrow without the help of any technical symbolism. The quantity which by its differentiation produces a proposed differential, is called the "integral" of this differential; since we consider it as having been formed by infinitely small continual additions: each of these additions is what we have named the differential of the increasing quantity, it is a fraction of it: and the sum of all these fractions is the entire quantity which we are in search of. For the same reason we call "integrating" or "taking the sum of" a differential the finding the integral of the sum of all the infinitely small successive additions which form the series, the differential of which, properly speaking, is the general term.

It is evident that two variables which constantly remain equal increase the one as much as the other during the same time, and that consequently their differences are equal: and the same holds good even if these two quantities had differed by any quantity whatever when they began to vary; provided that this primitive difference be always the same, their differentials will always be equal.
Reciprocally, it is clear that two variables which receive at each instant infinitely small equal additions must also either remain constantly equal to one another, or always differ by the same quantity—that is, the integrals of two differentials which are equal can only differ from each other by a constant quantity. For the same reason, if any two quantities whatever differ in an infinitely small degree from each other, their differentials will also differ from one another infinitely little: and reciprocally if two differential quantities differ infinitely little from one another, their integrals, putting aside the constant, can also differ but infinitely little one from the other.

Now, some of the rules for differentiation are as follows. If \( y = f(x) \), \( dy = f(x+dx) - f(x) \), in which higher powers of differentials added to lower ones may be neglected. Thus, if \( y = x^2 \), then \( dy = (x+dx)^2 - x^2 = 2x \cdot dx + (dx)^2 = 2x \cdot dx \). Here it is well to refer back to the treatment of the problem of tangents at the beginning of this chapter. Again, if \( y = a \cdot x \), where \( a \) is constant, \( dy = a \cdot dx \). If \( y = x \cdot z \), then \( dy = (x+dx)(z+dz) - x \cdot z = x \cdot dz + z \cdot dx \). If \( y = \frac{x}{z} \), \( x = y \cdot z \), so \( dx = y \cdot dz + z \cdot dy \); hence \( dy = \frac{dx - y \cdot dz}{z} \).

Since the integral calculus is the inverse of the differential calculus, we have at once

\[
\int 2x \cdot dx = x^2, \quad \int a \cdot dx = af \cdot dx, \\
\int x \cdot dz + \int z \cdot dx = xz,
\]

and so on. More fully, from \( d(x^2) = 3x^2 \cdot dx \), we conclude, not that \( \int x^2 \cdot dx = \frac{1}{3}x^3 \), but that \( \int x^2 \cdot dx = \frac{1}{3}x^3 + c \), where "c" denotes some constant depending on the fixed value for \( x \) from which the integration starts.

Consider a parabola \( y^2 = ax \); then \( 2y \cdot dy = a \cdot dx \).
or \( dx = \frac{2y \cdot dy}{a} \). Thus the area from the origin to the point \( x \) is \( \int \frac{2y^2 \cdot dy}{a} + c \); but \( \frac{2y^3}{3a} = \frac{2y^2 \cdot dy}{a} \); thus the area is \( \frac{2y^3}{3a} + c \), or, since \( y^2 = ax \), \( \frac{2}{3} x \cdot y + c \). To determine \( c \) when we measure the area from 0 to \( x \), we have the area zero when \( x = 0 \); hence the above equation gives \( c = 0 \). This whole result, now quite simple to us, is one of the greatest discoveries of Archimedes.

Let us now make a few short reflections on the infinitesimal calculus. First, the extraordinary power of it in dealing with complicated questions lies in that the question is split up into an infinity of simpler ones which can all be dealt with at once, thanks to the wonderfully economical fashion in which the calculus, like analytical geometry, deals with variables. Thus, a curvilinear area is split up into rectangular elements, all the rectangles are added together at once when it is observed that integral is the inverse of the easily acquired practice of differentiation. We must never lose sight of the fact that, when we differentiate \( y \) or integrate \( y \cdot dx \), we are considering, not a particular \( x \) or \( y \), but any one of an infinity of them. Secondly, we have seen that what in the first place had been regarded but as a simple method of approximation, leads at any rate in certain cases to results perfectly exact. The fact is that the exact results are due to a compensation of errors: the error resulting from the false supposition made, for example, by regarding a curve as a polygon with an infinite number of sides each infinitely small and which when produced is a tangent of the curve, is corrected or compensated for by that
which springs from the very processes of the calculus, according to which we retain in differentiation infinitely small quantities of the same order alone. In fact, after having introduced these quantities into the calculation to facilitate the expression of the conditions of the problem, and after having regarded them as absolutely zero in comparison with the proposed quantities, with a view to simplify these equations, in order to banish the errors that they had occasioned, and to obtain a result perfectly exact, there remains but to eliminate these same quantities from the equations where they may still be.

But all this cannot be regarded as a strict proof. There are great difficulties in trying to determine what infinitesimals are: at one time they are treated like finite numbers and at another like zeros or as "ghosts of departed quantities," as Bishop Berkeley, the philosopher, called them.

Another difficulty is given by differentials "of higher orders than the first." Let us take up again the considerations of the fourth chapter. We saw that \( v = \frac{d}{dt} s \), and found that \( s \) was got by integration: \( s = \int v \cdot dt \). This is now an immediate inference, since \( \frac{ds}{dt} dt = ds \). Now, let us substitute for \( v \) in \( \frac{dv}{dt} \).

Here \( t \) is the independent variable, and all of the older mathematicians treated the elements \( dt \) as constant—the interval of the independent variable was split up into atoms, so to speak, which themselves were regarded as known, and in terms of which other differentials, \( ds, dx, dy \), were to be determined. Thus

\[
\frac{dv}{dt} = \frac{d(dv)}{dt} = \frac{1}{dt} \cdot \frac{d(ds)}{dt} = \frac{d^2 s}{dt^2}
\]

"\( d^2 s \)" being written for "\( d(ds) \)" and "\( dt^2 \)" for
"(dt)^2". Thus the acceleration was expressed as "the second differential of the space divided by the square of \( dt \)." If \( \frac{d^2s}{dt^2} \) were constant, say, \( a \), then

\[
\frac{d^2s}{dt^2} = a \cdot dt;
\]

and, integrating both sides:

\[
\frac{ds}{dt} = \int a \cdot dt = a \int dt = at + b,
\]

where \( b \) is a new constant. Integrating again, we have:

\[
s = \int at \cdot dt + \int b \cdot dt = \frac{at^2}{2} + bt + c,
\]

which is a more general form of Galileo's result. Many complicated problems which show how far-reaching Galileo's principles are were devised by Leibniz and his school.

Thus, the infinitesimal calculus brought about a great advance in our powers of describing nature. And this advance was mainly due to Leibniz's notation; Leibniz himself attributed all of his mathematical discoveries to his improvements in notation. Those who know something of Leibniz's work know how conscious he was of the suggestive and economical value of a good notation. And the fact that we still use and appreciate Leibniz's "\( f \)" and "\( d \)" even though our views as to the principles of the calculus are very different from those of Leibniz and his school, is perhaps the best testimony to the importance of this question of notation. This fact that Leibniz's notations have become permanent is the great reason why I have dealt with his work before the analogous and prior work of Newton.

Isaac Newton (1642–1727) undoubtedly arrived at the principles and practice of a method equivalent to the infinitesimal calculus much earlier than
Leibniz, and, like Roberval, his conceptions were obtained from the dynamics of Galileo. He considered curves to be described by moving points. If we conceive a moving point as describing a curve, and the curve referred to co-ordinate axes, then the velocity of the moving point can be decomposed into two others parallel to the axes of \( x \) and \( y \) respectively; these velocities are called the "fluxions" of \( x \) and \( y \), and the velocity of the point is the fluxion of the arc. Reciprocally the arc is the "fluent" of the velocity with which it is described. From the given equation of the curve we may seek to determine the relations between the fluxions—and this is equivalent to Leibniz's problem of differentiation;—and reciprocally we may seek the relations between the co-ordinates when we know that between their fluxions, either alone or combined with the co-ordinates themselves. This is equivalent to Leibniz's general problem of integration, and is the problem to which we saw, at the end of the fourth chapter, that theoretical astronomy reduces.

Newton denoted the fluxion of \( x \) by "\( \dot{x} \)" and the fluxion of the fluxion (the acceleration) of \( \dot{x} \) by "\( \ddot{x} \)." It is obvious that this notation becomes awkward when we have to consider fluxions of higher orders; and further, Newton did not indicate by his notation the independent variable considered. Thus "\( \dot{y} \)" might possibly mean either \( \frac{dy}{dt} \) or \( \frac{dy}{dx} \). We have \( \dot{x} = \frac{dx}{dt} \), \( \ddot{x} = \frac{d^2x}{dt^2} \); but a dot-notation for \( \frac{d^n x}{dt^n} \) would be clumsy and inconvenient. Newton's notation for the "inverse method of fluxions" was far clumsier even, and far inferior to Leibniz's "\( f \)".

The relations between Newton and Leibniz were at
first friendly, and each communicated his discoveries to the other with a certain frankness. Later, a long and acrimonious dispute took place between Newton and Leibniz and their respective partisans. Each accused—unjustly, it seems—the other of plagiarism, and mean suspicions gave rise to meanness of conduct, and this conduct was also helped by what is sometimes called "patriotism." Thus, for considerably more than a century, British mathematicians failed to perceive the great superiority of Leibniz's notation. And thus, while the Swiss mathematicians, James Bernoulli (1654-1705), John Bernoulli (1667-1748), and Leonhard Euler (1707-1783), the French mathematicians d'Alembert (1707-1783), Clairaut (1713-1765), Lagrange (1736-1813), Laplace (1749-1827), Legendre (1752-1833), Fourier (1768-1830), and Poisson (1781-1850), and many other Continental mathematicians, were rapidly extending knowledge by using the infinitesimal calculus in all branches of pure and applied mathematics, in England comparatively little progress was made. In fact, it was not until the beginning of the nineteenth century that there was formed, at Cambridge, a Society to introduce and spread the use of Leibniz's notation among British mathematicians: to establish, as it was said, "the principles of pure d-ism in opposition to the dot-age of the university."

The difficulties met and not satisfactorily solved

* It is difficult for a mathematician not to think that the sudden and brilliant dawn on eighteenth century France of the magnificent and apparently all-embracing physics of Newton and the wonderfully powerful mathematical method of Leibniz inspired scientific men with the belief that the goal of all knowledge was nearly reached and a new era of knowledge quickly striding towards perfection begun; and that this optimism had indirectly much to do in preparing for the French Revolution.
by Newton, Leibniz, or their immediate successors, in the principles of the infinitesimal calculus, centre about the conception of a "limit"; and a great part of the meditations of modern mathematicians, such as the Frenchman Cauchy (1789–1857), the Norwegian Abel (1802–1829), and the German Weierstrass (1815–1897), not to speak of many still living, have been devoted to the putting of this conception on a sound logical basis.

We have seen that, if \( y = x^2 \), \( \frac{dy}{dx} = 2x \). What we do in forming \( \frac{dy}{dx} \) is to form \( \frac{(x + \Delta x)^2 - x^2}{\Delta x} \), which is readily found to be \( 2x + \Delta x \), and then consider that, as \( \Delta x \) approaches 0 more and more, the above quotient approaches \( 2x \). We express this by saying that the "limit, as \( h \) approaches 0," is \( 2x \). We do not consider \( \Delta x \) as being a fixed "infinitesimal" or as an absolute zero (which would make the above quotient become indeterminate \( \frac{0}{0} \)), nor need we suppose that the quotient reaches its limit (the state of \( \Delta x \) being 0). What we need to consider is that "\( \Delta x \)" should represent a variable which can take values differing from 0 by as little as we please. That is to say, if we choose any number, however small, there is a value which \( \Delta x \) can take, and which differs from 0 by less than that number. As before, when we speak of a "variable," we mean that we are considering a certain class. When we speak of a "limit," we are considering a certain infinite class. Thus the sequence of an infinity of terms 1, ½, \( \frac{1}{4} \), \( \frac{1}{8} \), \( \frac{1}{16} \), and so on, whose law of formation is easily seen, has the limit 0. In this case 0 is such that any number greater than it is greater than some term of the sequence, but 0 itself is not greater than any term of
the sequence and is not a term of the sequence. A sequence like $1, 1+\frac{1}{2}, 1+\frac{1}{2}+\frac{1}{4}, 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8} \ldots$, has an analogous upper limit 2. A function $f(x)$, as the independent variable $x$ approaches a certain value, like $\frac{2x}{x}$ as $x$ approaches 0, may have a value (in this case 2, though at 0, $\frac{2x}{x}$ is indeterminate). The question of the limits of a function in general is somewhat complicated, but the most important limit is $\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$ as $\Delta x$ approaches 0; this, if $y = f(x)$, is $\frac{dy}{dx}$.

That the infinitesimal calculus, with its rather obscure "infinitesimals"—treated like finite numbers when we write $\frac{dy}{dx} \cdot dx = dy$ and $\frac{1}{\frac{dy}{dx}} = \frac{dx}{dy}$, and then, on occasion, neglected—leads so often to correct results is a most remarkable fact, and a fact of which the true explanation only appeared when Cauchy, Gauss (1777–1855), Riemann (1826–1866), and Weierstrass had developed the theory of an extensive and much used class of functions. These functions happen to have properties which make them especially easy to be worked with, and nearly all the functions we habitually use in mathematical physics are of this class. A notable thing is that the complex numbers spoken of in the second chapter make this theory to a great extent.

Large tracts of mathematics have, of course, not been mentioned here. Thus, there is an elaborate theory of integer numbers to be referred to in a note to the seventh chapter, and a geometry using the
conceptions of the ancient Greeks and methods of modern mathematical thought; and very many men still regard space-perception as something mathematics deals with. We will return to this soon. Again, algebra has developed and branched off; the study of functions in general and in particular has grown; and soon a list of some of the many great men who have helped in all this would not be very useful. Let us now try to resume what we have seen of the development of mathematics along what seem to be its main lines.

In the earliest times men were occupied with particular questions—the properties of particular numbers and geometrical properties of particular figures, together with simple mechanical questions. With the Greeks, a more general study of classes of geometrical figures began. But traces of an earlier exception to this study of particulars are afforded by "algebra." In it and its later form symbols—like our present \( x \) and \( y \)—took the place of numbers, so that, what is a great advance in economy of thought and other labour, a part of calculation could be done with symbols instead of numbers, so that the one result stated, in a manner analogous to that of Greek geometry, a proposition valid for a whole infinite class of different numbers.

The great revolution in mathematical thought brought about by Descartes in 1637 grew out of the application of this general algebra to geometry by the very natural thought of substituting the numbers expressing the lengths of straight lines for those lines. Thus a point in a plane—for instance—is determined in position by two numbers \( x \) and \( y \), or co-ordinates. Now, as the point in question varies in position, \( x \) and \( y \) both vary; to every \( x \) belongs, in general, one or more \( y \)'s, and we arrive at the most beautiful idea
of a single algebraical equation between $x$ and $y$ representing the whole of a curve—the one "equation of the curve" expressing the general law by which, given any particular $x$ out of an infinity of them, the corresponding $y$ or $y$'s can be found.

The problem of drawing a tangent—the limiting position of a secant, when the two meeting points approach indefinitely close to one another—at any point of a curve came into prominence as a result of Descartes' work, and this, together with the allied conceptions of velocity and acceleration "at an instant," which appeared in Galileo's classical investigation, published in 1638, of the law according to which freely falling bodies move, gave rise at length to the powerful and convenient "infinitesimal calculus" of Leibniz and the "method of fluxions" of Newton. Mathematically, the finding of the tangent at the point of a curve, and finding the velocity of a particle describing this curve when it gets to that point, are identical problems. They are expressed as finding the "differential quotient," or the "fluxion" at the point. It is now known to be very probable that the above two methods, which are theoretically—but not practically—the same, were discovered independently; Newton discovered his first, and Leibniz published his first, in 1684. The finding of the areas of curves and of the shapes of the curves which moving particles describe under given forces showed themselves, in this calculus, as results of the inverse process to that of the direct process which serves to find tangents and the law of attraction to a given point from the datum of the path described by a particle. The direct process is called "differentiation," the inverse process "integration."

Newton's fame is chiefly owing to his application of this method to the solution, which, in its broad outlines, he gave of the problem of the motion of
the bodies in the solar system, which includes his discovery of the law according to which all matter gravitates towards—is attracted by—other matter. This was given in his *Principia* of 1687; and for more than a century afterwards mathematicians were occupied in extending and applying the calculus.

Then came more modern work, more and more directed towards the putting of mathematical methods on a sound logical basis, and the separation of mathematical processes from the sense-perception of space with which so much in mathematics grew and grows up. Thus trigonometry took its place by algebra as a study of certain mathematical functions, and it began to appear that the true business of geometry is to supply beautiful and suggestive pictures of abstract—"analytical" or "algebraical" or even "arithmetical," as they are called—processes of mathematics. In the next chapter we shall be concerned with part of the work of logical examination and reconstruction.