species. But it may happen that no \( P \) vanishes, and in this case all finite derivatives may have common points. The points which all have in common form a collection which is defined as \( P^\infty \). It is to be observed that \( P \) is thus defined without requiring the definition of \( \omega \). A term \( x \) belongs to \( P \) if, whatever finite integer \( v \) may be, \( x \) belongs to \( P^v \). It is to be observed that, though \( P \) may contain points not belonging to \( P \), yet subsequent derivatives introduce no new points. This illustrates the creative nature of the method of limits, or rather of segments: when it is first applied, it may yield new terms, but later applications give no further terms. That is, there is an intrinsic difference between a series which has been, or may have been, obtained as the derivative of some other series, and one not so obtainable. Every series which contains its first derivative itself is the derivative of an infinite number of other series*. The successive derivatives, like the segments determined by the various terms of a regression, form a series in which each term is part of each of its predecessors; hence \( P^\infty \), if it exists, is the lower limit of all the derivatives of finite order. From \( P \) it is easy to go on to \( P^{\infty+1}, P^{\infty+2}, \ldots \) etc. Series can be actually constructed in which any assigned derivative, finite or transfinite of the second class, is the first to vanish. When none of the finite derivatives vanishes, \( P \) is said to be of the second genus. It must not be inferred, however, that \( P \) is not denumerable. On the contrary, the first derivative of the rationals is the number-continuum, which is perfect, so that all its derivatives are identical with itself; yet the rationals, as we know, are denumerable. But when \( P^\infty \) vanishes, \( P \) is always denumerable, if \( x \) be finite or of the second class.

The theory of derivatives is of great importance to the theory of real functions†, where it practically enables us to extend mathematical induction to any ordinal of the second class. But for philosophy, it seems unnecessary to say more of it than is contained in the above remarks and in those of Chapter xxxvi. Popularly speaking, the first derivative consists of all points in whose neighborhood an infinite number of terms of the collection are heaped up; and subsequent derivatives give, as it were, different degrees of concentration in any neighborhood. Thus it is easy to see why derivatives are relevant to continuity: to be continuous, a collection must be as concentrated as possible in every neighborhood containing any terms of the collection. But such popular modes of expression are incapable of the precision which belongs to Cantor's terminology.

* Formulario de Mathematiques, Vol. ii, Part iii, § 71, 4-9.
† See Dini, Theorie der Functionen, Leipzig, 1899; esp. Chap. xii and Translator's preface.

CHAPTER XXXIX.

THE INFINITESIMAL CALCULUS.

303. The Infinitesimal Calculus is the traditional name for the differential and integral calculus together, and as such I have retained it; although, as we shall shortly see, there is no allusion to, or implication of, the infinitesimal in any part of this branch of mathematics.

The philosophical theory of the Calculus has been, ever since the subject was invented, in a somewhat disgraceful condition. Leibniz himself—who, one would have supposed, should have been competent to give a correct account of his own invention—and his ideas, upon this topic, which can only be described as extremely crude. He appears to have held that, if metaphysical subtleties are left aside, the Calculus is only approximate, but is justified practically by the fact that the errors to which it gives rise are less than those of observation*. When he was thinking of Dynamics, his belief in the actual infinitesimal hindered him from discovering that the Calculus rests on the doctrine of limits, and made him regard his \( dx \) and \( dy \) as neither zero, nor finite, nor mathematical fictions, but as really representing the units to which, in his philosophy, infinite division was supposed to lead‡. And in his mathematical expositions of the subject, he avoided giving careful proofs, confining himself with the enumeration of rules§. At other times, it is true, he definitely rejects infinitesimals as philosophically valid‖; but he failed to show how, without the use of infinitesimals, the results obtained by means of the Calculus could yet be exact, and not approximate. In this respect, Newton is preferable to Leibniz: his Leibniz‖ give the true foundation of the Calculus in the doctrine of limits, and, assuming the continuity of space and time in Cantor's sense, they give valid proofs

‖ See Principia, Part I, Section 1.
of its rules so far as spatio-temporal magnitudes are concerned. But Newton was, of course, entirely ignorant of the fact that his Leibniz' depend upon the modern theory of continuity; moreover, the appeal to time and change, which appears in the word fluxion, and to space, which appears in the Leibniz, was wholly unnecessary, and served merely to hide the fact that no definition of continuity had been given. Whether Leibniz avoided this error, seems highly doubtful: it is at any rate certain that, in his first published account of the Calculus, he defined the differential coefficient by means of the tangent to a curve. And by his emphasis on the infinitesimal, he gave a wrong direction to speculation as to the Calculus, which misled all mathematicians before Weierstrass (with the exception, perhaps, of De Morgan), and all philosophers down to the present day. It is only in the last thirty or forty years that mathematicians have provided the requisite mathematical foundations for a philosophy of the Calculus; and these foundations, as is natural, are as yet little known among philosophers, except in France. Philosophical works on the subject, such as Cohen's *Principi der Infinitesimalmethode und seine Geschichte*, are vitiated, as regards the constructive theory, by an undue mysticism, inherited from Kant, and leading to such results as the identification of intensive magnitude with the extensive infinitesimal. I shall examine in the next chapter the conception of the infinitesimal, which is essential to all philosophical theories of the Calculus hitherto propounded. For the present, I am only concerned to give the constructive theory as it results from modern mathematics.

304. The differential coefficient depends essentially upon the notion of a continuous function of a continuous variable. The notion to be defined is not purely ordinal; on the contrary, it is applicable, in the first instance, only to series of numbers, and thence, by extension, to series in which distances or stretches are numerically measurable. But first of all we must define a continuous function.

We have already seen (Chap. xxxvii.) what is meant by a function of a variable, and what is meant by a continuous variable (Chap. xxxvi.). If the function is one-valued, and is only ordered by correlation with the variable, then, when the variable is continuous, there is no sense in asking whether the function is continuous; for such a series by correlation is always always ordinally similar to its prototype. But when, as where the variable and the field of the function are both classes of numbers, the function has an order independent of correlation, it may or may not happen that the values of the function, in the order obtained by correlation, form a continuous series in the independent order. When they do so in any interval, the function is said to be continuous in that interval. The

precise definitions of continuous and discontinuous functions, when $x$ and $f(x)$ are numerical, are given by Dirichlet as follows. The dependent variable $x$ is considered to consist of the real numbers, or the real numbers in a certain interval; $f(x)$, in the interval case, is to be one-valued, even at the end-points of the interval, and is also composed of real numbers. We then have the following definitions:

"We call $f(x)$ continuous for $x = a$, or in the point a, if we have the value $f(a)$, if for every positive number $\epsilon$, defined but as small as we please, there exists a positive number $\delta$ from 0, such that, for all values of $\delta$ which are numerically less than $\delta$, the difference $f(a + \delta) - f(a)$ is numerically less than $\epsilon$. In other words, $f(x)$ is continuous in the point $x = a$, where it has the value $f(a)$, if the limit of its values to the right and left of $a$ is the same equal to $f(a)$.

"Again, $f(x)$ is discontinuous for $x = a$, if, for any positive $\epsilon$, there is no corresponding positive value of $\delta$ such that, for all values of $\delta$ which are numerically less than $\delta$, the difference $f(a + \delta) - f(a)$ is less than $\epsilon$; in other words, $f(x)$ is discontinuous for $x = a$, if for any values $f(x + h)$ of $f(x)$ to the right of $a$, and the values $f(x - h)$ to the left of $a$, the one and the other, have no determinate limit they have such, these are different on the two sides of $a$; or, if the same, they differ from the value $f(a)$, which the function has at point $a$.

These definitions of the continuity and discontinuity of a function must be confessed, are somewhat complicated; but it seems impossible to introduce any simplification without loss of rigour. Roughly, say that a function is continuous in the neighbourhood of $a$, values $f(x)$ as it approaches $a$ approach the value $f(a)$, and have their limit both to left and right. But the notion of the limit function is somewhat more complicated notion than that of a general, with which we have been hitherto concerned. A function perfectly general kind will have no limit as it approaches a point. In order that it should have a limit as $x$ approaches $a$, it is necessary and sufficient that, if any number $\epsilon$ be such that any two values of $f(x)$, when $x$ is sufficiently near to $a$, but less differ by less than $\epsilon$; in popular language, the value of the does not make any sudden jumps as $x$ approaches $a$. Under similar circumstances, $f(x)$ will have a limit as it approaches from the right. But these limits, when both exist, are equal either to each other, or to $f(a)$, the value of the function.
In order that the values of \( f(x) \) to the right or left of a finite number \( a \) (for instance to the right) should have a determinate and finite limit, it is necessary and sufficient that, for every arbitrarily small positive number \( \varepsilon \), there should be a positive number \( \delta \), such that the difference \( f(x + \delta) - f(x) \) for \( x = x + \varepsilon \), and the value \( f(x + \delta) \), which corresponds to the value \( x + \delta \) of \( x \), should be numerically less than \( \varepsilon \), for every \( \delta \) which is greater than 0 and less than \( \varepsilon \).

It is possible, instead of this, to define a limit of a function, and then discuss whether it exists, to define generally a whole class of limits. In this method, a number \( x \) belongs to the class of limits of \( f(x) \) for \( x = a \), if, within any interval containing \( a \), however small, \( y \) will approach nearer to \( x \) than by any given difference. Thus, for example, \( \sin x / x \), as \( x \) approaches zero, will take every value from 1 to 0 (both inclusive) in every finite interval containing zero, however small, \( y \) will approach nearer to \( x \) than by any given difference. Thus, in fact, \( \sin x / x \) is defined as a function of \( \delta \), which is indeterminate when \( \delta = 0 \).

The limit of a function for a given value of the independent variable is, as we have seen, an entirely different notion from its value for the said value of the independent variable, and the limit may or may not be the same number. In the present case, the limit may be definite, but the value for \( \delta = 0 \) may have no meaning. Thus it is the doctrine of limits that underlies the Calculus, and not any pretended use of the infinitesimal. This is the only point of philosophic importance in the present subject, and it is only to elicit this point that I have dragged the reader through so much mathematics.

Just as the derivative of a function is the limit of a fraction, so the definite integral is the limit of a sum. The definite integral may be defined as follows:

\[
\int_a^b f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x
\]

where \( x_i \) are the points of division of the interval \( [a, b] \) and \( \Delta x = (b - a) / n \). The definition of the definite integral differs little in different modern works.

* Dini, op. cit. p. 38.
† See Peano, "Ristitui di Matematica," pp. 77-79; Formulario, Part III, § 74. 10.
may be chosen in its interval, and however the intervals be chosen (provided only that all are less than any assigned number for sufficiently great values of \(a\))—then this one limit is called the definite integral of \(f(x)\) from \(a\) to \(\beta\). If there is no such limit, \(f(x)\) is not integrable from \(a\) to \(\beta\).

308. As in the case of the derivative, there is only one important remark to make about this definition. The definite integral involves neither the infinite nor the infinitesimal, and is itself not a sum, but only and strictly the limit of a sum. All the terms which occur in the sum whose limit is the definite integral are finite, and the sum itself is finite. If we were to suppose the limit actually attained, it is true, the number of intervals would be infinite, and the magnitude of each would be infinitesimal; but in this case, the sum becomes meaningless.

Thus the sum must not be regarded as actually attaining its limit. But this is a respect in which series in general agree. Any series which always ascends or always descends and has no last term cannot reach its limit; other infinite series may have a term equal to their limit, but if so, this is a mere accident. The general rule is that the limit does not belong to the series which it limits; and in the definition of the derivative, and the definite integral we have merely another instance of this fact.

The so-called infinitesimal calculus, therefore, has nothing to do with the infinitesimal, and has only indirectly to do with the infinite—its connection with the infinite being, that it involves limits, and only infinite series have limits.

The above definitions, since they involve multiplication and division, are essentially arithmetical. Unlike the definitions of limits and continuity, they cannot be rendered purely ordinal. But it is evident that they may be at once extended to any numerically measurable magnitudes, and therefore to all series in which stretches or distances can be measured. Since spaces, times, and motions are included under this head, the Calculus is applicable to Geometry and Dynamics. As to the axioms involved in the assumption that geometrical and dynamical functions can be differentiated and integrated, I shall have something to say at a later stage. For the present, it is time to make a critical examination of the infinitesimal on its own account.

CHAPTER XL.

THE INFINITESIMAL AND THE IMPROPER INFINITE.

309. Until recent times, it was universally believed that continuity, the derivative, and the definite integral, all involved actual infinitesimals, i.e. that even if the definitions of these notions could be formally freed from explicit mention of the infinitesimal, yet, where the definitions applied, the actual infinitesimal must always be found. This belief is now generally abandoned. The definitions which have been given in previous chapters do not in any way imply the infinitesimal, and this notion appears to have become mathematically useless. In the present chapter, I shall first give a definition of the infinitesimal, and then examine the cases where this notion arises. I shall end by a critical discussion of the belief that continuity implies the infinitesimal.

The infinitesimal has, in general, been very vaguely defined. It has been regarded as a number or magnitude which, though not zero, is less than any finite number or magnitude. It has been the \(dx\) or \(dy\) of the Calculus, the time during which a ball thrown vertically upwards is at rest at the highest point of its course, the distance between a point on a line and the next point, etc., etc. But none of these notions are at all precise. The \(dx\) and \(dy\), as we saw in the last chapter, are nothing at all: \(dy/dx\) is the limit of a fraction whose numerator and denominator are finite, but is not itself a fraction at all. The time during which a ball is at rest at its highest point is a very complex notion, involving the whole philosophical theory of motion: in Part VII we shall find, when this theory has been developed, that there is no such time. The distance between consecutive points presupposes that there are consecutive points—a view which there is every reason to deny. And so with most instances—they afford no precise definition of what is meant by the infinitesimal.

310. There is, so far as I know, only one precise definition, which renders the infinitesimal a purely relative notion, correlative to something arbitrarily assumed to be finite. When, instead, we regard what had been taken to be infinitesimal as finite, the correlative notion is what Cantor calls the improper infinite (Unendlich-Unendlicher). The
an infinite number. The number of parts being taken as the measure, every infinite whole will be greater than \( n \) times every finite whole, whatever finite number \( n \) may be. This is therefore a perfectly clear instance. But it must not be supposed that the ratio of the divisibilities of two wholes, of which one at least is transfinite, can be measured by the ratio of the cardinal numbers of their simple parts. There are two reasons why this cannot be done. The first is, that two transfinite cardinals do not have any relation strictly analogous to ratio; indeed, the definition of ratio is effected by means of mathematical induction. The relation of two transfinite cardinals \( \alpha, \gamma \) expressed by the equation \( \alpha \beta = \gamma \delta \) bears a certain resemblance to integral ratios, and \( \alpha \beta = \gamma \delta \) may be used to define other ratios. But ratios so defined are not very similar to finite ratios. The other reason why infinite divisibilities must not be measured by transfinite numbers is, that the whole must always have more divisibility than the part (provided the remaining part is not relatively infinitesimal), though it may have the same transfinite number. In short, divisibilities, like ordinals, are equal, so long as the wholes are finite, when and only when the cardinal numbers of the wholes are the same; but the notion of magnitude of divisibility is distinct from that of cardinal number, and separates itself visibly as soon as we come to infinite wholes.

Two infinite wholes may be such that one is infinitely less divisible than the other. Consider, for example, the length of a finite straight line and the area of the square upon that straight line; or the length of a finite straight line and the length of the whole straight line of which it forms part (except in finite spaces); or an area and a volume; or the rational numbers and the real numbers; or the collection of points on a finite part of a line obtainable by von Staudt’s quadrilateral construction, and the total collection of points on the said finite part*. All these are magnitudes of one and the same kind, namely divisibilities, and all are infinite divisibilities; but they are of many different orders. The points on a limited portion of a line obtainable by the quadrilateral construction form a collection which is infinitesimal with respect to the said portion; this portion is ordinarily infinitesimal† with respect to any bounded area; any bounded area is ordinarily infinitesimal with respect to any bounded volume; and any bounded volume (except in finite spaces) is ordinarily infinitesimal with respect to all space. In all these cases, the word infinitesimal is used strictly according to the above definition, obtained from the axiom of Archimedes. What makes these various infinitesimals somewhat unimportant, from a mathematical standpoint, is, that measurement essentially depends upon the axiom of Archimedes, and cannot, in general, be extended by means of transfinite numbers, for the reasons which have just been explained. Hence two divisibilities, of

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* See Part VI, Chap. xlv.  † See Part VI, Chap. xlvii, § 337.
which one is infinitesimal with respect to the other, are regarded usually as different kinds of magnitude; and to regard them as of the same kind gives no advantage save philosophic correctness. All of them, however, are strictly instances of infinitesimals, and the series of them well illustrates the relativity of the term infinitesimal.

An interesting method of comparing certain magnitudes, analogous to the divisibilities of any infinite collections of points, with those of continuous stretches is given by Stolz*, and a very similar but more general method is given by Cantor†. These methods are too mathematical to be fully explained here, but the gist of Stolz's method may briefly be explained. Let a collection of points $x'$ be contained in some finite interval $a$ to $b$. Divide the interval into any number $n$ of parts, and divide each of these parts again into any number of parts, and so on; and let the successive divisions be so effected that all parts become in time less than any assigned number $\delta$. At each stage, add together all the parts that contain points of $x'$. At the $n$th stage, let the resulting sum be $S_n$. Then successive divisions may diminish this sum, but cannot increase it. Hence as the number of divisions increases, $S_n$ must approach a limit $L$. If $x'$ is compact throughout the interval, we shall have $L = b - a$; if any finite derivative of $x'$ vanishes, $L = 0$. $L$ obviously bears an analogy to a definite integral; but no conditions are required for the existence of $L$. But $L$ cannot be identified with the divisibility; for some compact series, e.g. that of rationals, are less divisible than others, e.g. the continuum, but give the same value of $L$.

312. The case in which infinitesimals were formerly supposed to be peculiarly evident is that of compact series. In this case, however, it is possible to prove that there can be no infinitesimal segments, provided numerical measurement is possible at all—and if it be not possible, the infinitesimal, as we have seen, is not definable. In the first place, it is evident that the segment contained between two different terms is always infinitely divisible; for since there is a term $c$ between any two $a$ and $b$, there is another $d$ between $a$ and $c$, and so on. Thus no terminated segment can contain a finite number of terms. But segments defined by a class of terms may (as we saw in Chapter xxxiv) have no limiting term. In this case, however, provided the segment does not consist of a single term $a$, it will contain some other term $b$, and therefore an infinite number of terms. Thus all segments are infinitely divisible. The next point is to define multiples of segments. Two terminated segments can be added by placing a segment equal to the one at the end of the other to form a new segment; and if the two were equal, the new one is said to be double of each of them. But if the two segments are not terminated, this process cannot be employed. Their sum, in this case, is defined by Professor Peano as the logical sum of all the segments obtained by adding two terminated segments contained respectively in the two segments to be added*. Having defined this sum, we can define any finite multiple of a segment. Hence we can define the class of terms contained in some finite multiple of our segment, i.e. the logical sum of all its finite multiples. If, with respect to all greater segments, our segment obeys the axiom of Archimedes, then this new class will contain all terms that come after the origin of our segment. But if our segment be infinitesimal with respect to any other segment, then the class in question will fail to contain some points of this other segment. In this case, it is shown that all transfinite multiples of our segment are equal to each other. Hence it follows that the class formed by the logical sum of all finite multiples of our segment, which may be called the infinite multiple of our segment, must be a non-terminated segment, for a terminated segment is always increased by being doubled. "Each of these results," so Professor Peano concludes, "is in contradiction with the usual notion of a segment. And from the fact that the infinitesimal segment cannot be rendered finite by means of any actually infinite multiplication, I conclude, with Cantor, that it cannot be an element in finite magnitudes" (p. 69). But I think an even stronger conclusion is warranted; for we have seen that, in compact series, there is, corresponding to every segment, a segment of segments, and this is always terminated by its defining segment; further that the numerical measurement of segments of segments is exactly the same as that of simple segments; whence, by applying the above result to segments of segments, we obtain a definite contradiction, since none of them can be unterminated, and an infinitesimal one cannot be terminated.

In the case of the rational or the real numbers, the complete knowledge which we possess concerning them renders the non-existence of infinitesimals demonstrable. A rational number is the ratio of two finite integers, and any such ratio is finite. A real number other than zero is a segment of the series of rationals; hence if $x$ be a real number other than zero, there is a class $u$, not null, of rationals such that, if $y$ is a $u$, and $z$ is less than $y$, $z$ is an $x$, i.e. belongs to the segment which is $x$. Hence every real number other than zero is a class containing rationals, and all rationals are finite; consequently every real number is finite. Consequently if it were possible, in any sense, to speak of infinitesimal numbers, it would have to be in some radically new sense.

313. I come now to a very difficult question, on which I would gladly say nothing—I mean, the question of the orders of infinity and infinitesimality of functions. On this question the greatest authorities

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are divided: Du Bois Reymond, Stolz, and many others, maintaining that these form a special class of magnitudes, in which actual infinitesimals occur, while Cantor holds strongly that the whole theory is erroneous. To put the matter as simply as possible, consider a function \( f(x) \) whose limit, as \( x \) approaches zero, is zero. It may happen that, for some finite real number \( a \), the ratio \( f(x)/x \) has a finite limit as \( x \) approaches zero. There can be only one such number, but there may be none. Then, if there is such a number, may be called the order to which \( f(x) \) becomes infinitesimal, or the order of smallness of \( f(x) \) as \( x \) approaches zero. But for some functions, e.g. \( \log x \), there is no such number \( a \). If \( a \) be any finite real number, the limit of \( 1/x^a \log x \), as \( x \) approaches zero, is infinite. That is, when \( x \) is sufficiently small, \( 1/x^a \log x \) is very large, and may be made larger than any assigned number by making \( x \) sufficiently small—and this whatever finite number \( a \) may be. Hence, to express the order of smallness of \( 1/\log x \), it is necessary to invent a new infinitesimal number, which may be denoted by \( 1/g \). Similarly we shall need infinitely great numbers to express the order of smallness of \( (\text{say}) e^{1/x} \) as \( x \) approaches zero. And there is no end to the succession of these orders of smallness: that of \( 1/\log (\log x) \), for example, is infinitely smaller than that of \( 1/\log x \), and so on. Thus we have a whole hierarchy of magnitudes, of which all in any one class are infinitesimal with respect to all in any higher class, and of which one class only is formed of all the finite real numbers.

In this development, Cantor finds a vicious circle; and though the question is difficult, it would seem that Cantor is in the right. He objects (loc. cit.) that such magnitudes cannot be introduced unless we have reason to think that there are such magnitudes. The point is similar to that concerning limits; and Cantor maintains that, in the present case, definite contradictions may be proved concerning the supposed infinitesimals. If there were infinitesimal numbers \( j \), then even for them we should have

\[ \lim_{x \to 0} \frac{1}{(\log x \cdot x^0)} = 0 \]

since \( x^0 \) must ultimately exceed \( j \). And he shows that even continuous, differentiable, and uniformly growing functions may have an entirely ambiguous order of smallness or infinity: that, in fact, for some such functions, this order oscillates between infinite and infinitesimal values, according to the manner in which the limit is approached. Hence we may, I think, conclude that these infinitesimals are mathematical fictions. And this may be reinforced by the consideration that, if there were infinitesimal numbers, there would be infinitesimal segments of the number-continuum, which we have just seen to be impossible.


313, 314. Thus to sum up what has been said concerning the infinitesimal, we see, to begin with, that it is a relative term, and that, as regards magnitudes other than divisibilities, or divisibilities of wholes which are infinite in the absolute sense, it is not capable of being other than a relative term. But where it has an absolute meaning, there this meaning is indistinguishable from finitude. We saw that the infinitesimal, though completely useless in mathematics, does occur in certain instances—for example, lengths of bounded straight lines are infinitesimal as compared to areas of polygons, and these again as compared to volumes of polyhedra. But such genuine cases of infinitesimals, as we saw, are always regarded by mathematics as magnitudes of another kind, because no numerical comparison is possible, even by means of transfinite numbers, between an area and a length, or a volume and an area. Numerical measurement, in fact, is wholly dependent upon the axiom of Archimedes, and cannot be extended as Cantor has extended numbers. And finally we saw that there are no infinitesimal segments in compact series, and—what is closely connected—that orders of smallness of functions are not to be regarded as genuine infinitesimals. The infinitesimal, therefore—so we may conclude—is a very restricted and mathematically very unimportant conception, of which infinity and continuity are alike independent.